Physical Degrees of Freedom of Non-local Theories

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Abstract

We analyze the physical reduced space of non-local theories, around the fixed points of these systems, by analyzing: i) the Hamiltonian constraints appearing in the 1+1 formulation, ii) the symplectic two form in the surface on constraints.

P-adic string theory for spatially homogeneous configurations has two fixed points. The physical phase space around $q = 0$ is trivial, instead around $q = \frac{1}{g}$ is infinite dimensional. For the special case of the rolling tachyon solutions it is an infinite dimensional lagrangian submanifold. In the case of string field theory, at lowest truncation level, the physical phase space of spatially homogeneous configurations is two dimensional around $q = 0$, which is the relevant case for the rolling tachyon solutions, and infinite dimensional around $q = \frac{M^2}{g}$.

1 Introduction

Recently it has been a lot of interest to study open string tachyon condensation. There are several approaches to study this problem like string field theory, p-adic string theory, boundary conformal field theory, Born-Infeld effective theory, non-critical string theory, matrix models, see for example [1],[2],[3],[4],[5],[6],[7],[8],[9],[10]. In ref [4] the rolling of the tachyon has been analyzed by constructing a classical time dependent spatially homogeneous solution of p-adic string theory (p-adic particle). This solution has oscillations with time and ever-growing amplitude $\phi(t) = \sum_n a_n e^{nt}$.

Since the string field theory [11] and the p-adic string theory [12] are non-local theories the construction of solutions and the initial value problem, which is related to the dimension of the physical reduced phase space, are non-trivial issues.[1] Some aspects of non-local theories in connection with the string filed theory have been examined [13].

[1]A non-singular Lagrangian system up to n-th time derivatives has $2n$ degrees of freedom. The canonical description was given by Ostrogradski [14]. In general non-local systems have infinite degrees of freedom and their energies are not bounded. Naively in order to get solutions, in a generic point, one needs an infinite number of initial conditions.

In this paper, motivated by these problems, we present a general method to analyze reduced phase space of non-local theories. It is based on the 1+1 dimensional Hamiltonian formalism of non-local theories proposed in [15], and further developed in [16]. The formalism consists of a two dimensional field theory Hamiltonian and two sets of phase space constraints, momentum and Euler Lagrange (EL) constraints. In this framework the Euler Lagrange equation of motion appears as a Hamiltonian constraint. The reduced phase space is constructed by analyzing the first or second class character of these constraints. The analysis is analogous to the construction of the physical phase space of gauge theories. As we will see there are two types of reduced phase spaces around any fixed point that we call perturbative and non-perturbative. In the first case we solve the second class constraints in a smooth way for $g \to 0$, where g is a coupling constant of the non-local theory, and in the second case we solve non-perturbatively the second class constraints. An alternative way to analyze the structure of the reduced phase space consists in computing the infinite dimensional symplectic two form on the surface defined by the momentum and EL constraints, around the fixed points.

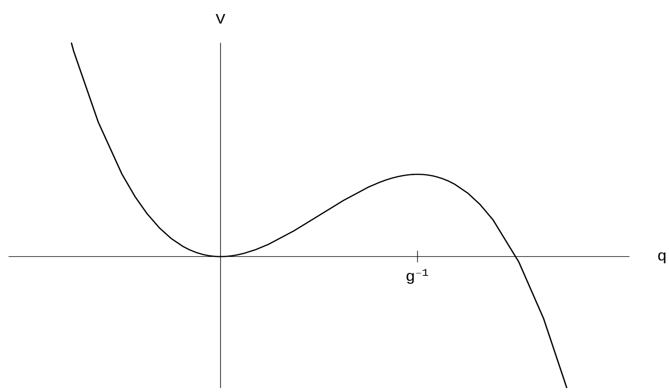

Figure 1: *Potential of P-adic Theory.* The phase space is zero dimensional at the minimum $q = 0$. It is infinite dimensional at the local maximum $q = 1/g$. For the rolling solutions it is infinite dimensional lagrangian submanifold.

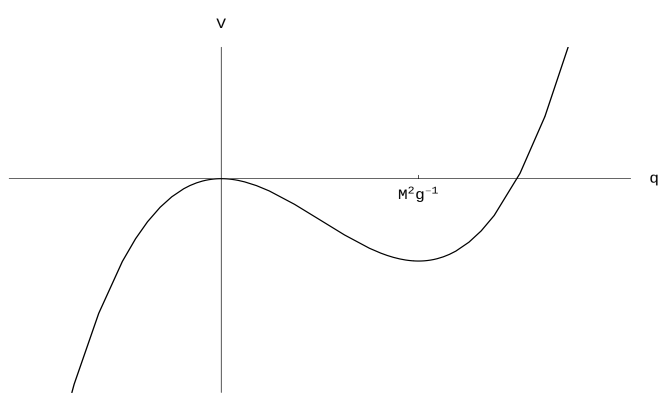

Figure 2: *Potential of String Field Theory.* The phase space is two dimensional at the local maximum $q = 0$. It is infinite dimensional at the local minimum $q = M^2/g$.

In the case of smooth spatially homogeneous configurations of p-adic string theory, we will see that the perturbative physical reduced phase space around the trivial

fixed point $q = 0$ is zero dimensional, where all the coordinates and momenta vanish (fig.1). There is no mode excitation around this minimum. For the fixed point $q = \frac{1}{g}$ the perturbative reduced phase space is infinite dimensional. The unstable and stable manifold of this non-trivial fixed point is a lagrangian submanifold of infinite dimensions, where the symplectic form vanishes. This implies that the manifold of solutions relevant for the rolling tachyon, unstable submanifold, is infinite dimensional without a phase space structure. If we do not impose the boundary condition, solutions do not have a definite sign of the energy. This is a general property for a generic situation in non-local theories.

We will also analyze the zero-th level truncation of string field theory as a non-local theory[13]. In this case the perturbative reduced phase space around $q = 0$ is two dimensional (fig.2). This implies that the space of solutions relevant for the rolling tachyon is two dimensional. The perturbative phase space around the fixed point $q = \frac{M^2}{g}$ is infinite dimensional. This implies that there are infinite excitation modes around this minimum. The unstable and stable manifold of this non-trivial fixed point is a lagrangian submanifold of infinite dimension where the symplectic form vanishes. The difference with respect to the p-adic case is due to the different dispersion relation for the solutions we found.

The non-local harmonic oscillator is examined to illustrate the formalism. It have been studied in detail in [13] and [15] and is known to be a finite dimensional system. Around the only one fixed point $q = 0$ we reproduce that the phase space is two dimensional perturbatively and four dimensional non-perturbatively. In these cases we will compute the energy for the set of solutions verifying the EL equations.

In section 2 a brief review of the 1+1 dimensional Hamiltonian formalism for non-local theories is introduced. In section 3 we apply it to a non-local harmonic oscillator and show how the infinite dimensional phase space is reduced using the constraints. In sections 4 and 5 we discuss the p-adic and string field theories. Discussions will be in the last section.

2 1+1 Dimensional Formalism of Non-local Theories

Ordinary local Lagrangians depend on a finite number of derivatives at a given time, it is

$$L(q(t), \dot{q}(t), \cdots, q^{(n)}(t)). \tag{1}$$

Now, we can consider a Lagrangian depending on a piece of the trajectory $q(t + \lambda)$, $\forall\lambda$ belonging to an interval $[a, b]$, where a and b are real numbers

$$L(q(t + \lambda)). \tag{2}$$

This is how we obtain a non-local Lagrangian , that we denoted by $L^{non}(t)$.

For sufficiently smooth trajectories we can expand $q(t + \lambda)$ to all powers in λ. In this case, this Lagrangian can be written as a function of all time derivatives $\frac{d^m}{dt^m}q(t)$ with $m = 0, 1, 2, \cdots$. The EL equation is

$$\int dt \frac{\delta L^{non}(t)}{\delta q(t')} = 0. \tag{3}$$

This equation should be understood as a functional relation to be satisfied by physical trajectories, i.e., a Lagrangian constraint. It defines a subspace J_R of physical trajectories

$$J_R \subset J \tag{4}$$

in the space of all possible trajectories J. There is no dynamics except the displacement inside the trajectory

$$q(t) \stackrel{T_\lambda}{\to} q(t+\lambda). \tag{5}$$

The $1+1$ dimensional formalism of non-local Lagrangians [15],[16] is based in considering a field $Q(t,\lambda)$, instead of trajectory $q(t)$, with the restriction [2]

$$\dot{Q}(t,\lambda) = Q'(t,\lambda) \tag{6}$$

where $\dot{Q}(t,\lambda) = \frac{\partial}{\partial t} Q(t,\lambda)$ and $Q'(t,\lambda) = \frac{\partial}{\partial \lambda} Q(t,\lambda)$. Equation (6) implies $Q(t,\lambda) = F(t+\lambda)$, we further assume $F(t) = q(t)$. The Hamiltonian is given by

$$H(t,[Q,P]) = \int d\lambda P(t,\lambda) Q'(t,\lambda) - \mathcal{L}(t,0), \tag{7}$$

where $P(t,\lambda)$ is the canonical momentum of $Q(t,\lambda)$. The phase space is thus T^*J with the Poisson brackets

$$\{Q(t,\lambda), P(t,\lambda')\} = \delta(\lambda - \lambda'). \tag{8}$$

The symplectic two form Ω is given by

$$\Omega = \int d\lambda\; dP(t,\lambda) \wedge dQ(t,\lambda). \tag{9}$$

The "Lagrangian density" $\mathcal{L}(t,\lambda)$ is constructed from the original non-local Lagrangian $L^{\rm non}(t)$ by replacing $q(t)$ by $Q(t,\lambda)$, t-derivatives of $q(t)$ by λ-derivatives of $Q(t,\lambda)$ and $q(t+\rho)$ by $Q(t,\lambda+\rho)$.

There are two sets of constraints in the formalism. One is associated with the definition of the momentum $P(t,\lambda)$, which for a non-local theory becomes a Hamiltonian constraint (momentum constraint)

$$\varphi(t,\lambda,[Q,P]) \equiv P(t,\lambda) - \int d\sigma\, \chi(\lambda,-\sigma)\, \mathcal{E}(t;\sigma,\lambda) \;\approx 0\,, \tag{10}$$

where $\mathcal{E}(t;\sigma,\lambda)$ and $\chi(\lambda,-\sigma)$ are defined by

$$\mathcal{E}(t;\sigma,\lambda) \;=\; \frac{\delta \mathcal{L}(t,\sigma)}{\delta Q(t,\lambda)}, \quad \chi(\lambda,-\sigma) \;=\; \frac{\epsilon(\lambda)-\epsilon(\sigma)}{2}. \tag{11}$$

The other is equivalent to the Euler Lagrange equation, (EL constraint), explicitly

$$\psi(t,\lambda,[Q]) \equiv \int d\sigma\; \mathcal{E}(t;\sigma,\lambda) \approx 0\,. \tag{12}$$

In fact if we use (6), $\psi \approx 0$ implies the EL equation of the original non-local Lagrangian $L^{\rm non}(t)$.

[2] An equivalent formalism was also given in [17] for a nonlocal system of finite extend.

The momentum constraints (10) are stable by using the EL constraints (12). The stability of the EL constraints is guaranteed by itself,

$$\dot{\varphi} \;=\; \{\varphi, H\} \;\sim\; \psi \;\approx\; 0, \qquad \dot{\psi} \;=\; \{\psi, H\} \;\sim\; \psi \;\approx\; 0. \tag{13}$$

Therefore the Hamiltonian (7), the momentum constraint (10) and the EL constraint (12) define a consistent Hamiltonian system [18].

It is local with respect to the "time" t although it appears non-locality with respect to the "space" λ. The momentum constraint (10) and the EL constraint (12) define a surface Σ^* in the T^*J.

In order to analyze the reduced phase space for smooth trajectories, it is convenient to make an expansion of the coordinate and momentum as

$$Q(t,\lambda) \;=\; \sum_{m=0}^{\infty} e_m(\lambda)\; q^m(t), \qquad P(t,\lambda) \;=\; \sum_{m=0}^{\infty} e^m(\lambda)\; p_m(t), \tag{14}$$

where $e_m(\lambda)$ and $e^m(\lambda)$ are ortho-normal basis

$$e_m(\lambda) \;=\; \frac{\lambda^m}{m!}, \qquad e^m(\lambda) \;=\; (-\partial_\lambda)^m \delta(\lambda), \tag{15}$$

$$\int d\lambda\; e^m(\lambda)\; e_\ell(\lambda) \;=\; \delta^m{}_\ell, \qquad \sum_{m=0}^{\infty} e^m(\lambda)\; e_m(\lambda') \;=\; \delta(\lambda-\lambda'). \tag{16}$$

The Poisson brackets for the variables q^m, p_n are given by

$$\{q^m, p_n\} = \delta^m{}_n. \tag{17}$$

The symplectic two form (9) is given by

$$\Omega = \sum_{m=0}^{\infty} dp_m \wedge dq^m. \tag{18}$$

The Hamiltonian (7) can be written as

$$H(q,p) = \sum_{m=0}^{\infty} p_m q^{m+1} - L(q^0, q^1, ...). \tag{19}$$

One of the Hamilton's equation implies $q^{m+1} = \frac{d}{dt}q^m$. The momentum constraint $\varphi(t,\lambda)$ can be expanded in terms of $e^m(\lambda)$ as

$$\varphi(t,\lambda) = \sum_{m=0}^{\infty}\; \varphi_m(t) e^m(\lambda) \tag{20}$$

and the EL constraint $\psi(t,\lambda)$ can be expanded by $e_m(\lambda)$ as

$$\psi(t,\lambda) = \sum_{m=0}^{\infty}\; \psi^m(t) e_m(\lambda). \tag{21}$$

Our formalism applied to local Lagrangians depending on finite number of time derivatives reproduces the Ostrogradski formalism [14].

2.1 Reduced phase space

In order to examine the reduced phase space and find the physical degrees of freedom, it is crucial to analyze the first and second class character of the constraints, the rank of the matrix of the Poisson brackets

$$\begin{pmatrix} \{\varphi(\lambda), \varphi(\lambda')\}, & \{\varphi(\lambda), \psi(\lambda')\} \\ \{\psi(\lambda), \varphi(\lambda')\}, & \{\psi(\lambda), \psi(\lambda')\} \end{pmatrix}, \tag{22}$$

or equivalently in terms of φ_m (20) and ψ^m (21),

$$\begin{pmatrix} \{\varphi_m, \varphi_\ell\}, & \{\varphi_m, \psi^\ell\} \\ \{\psi^m, \varphi_\ell\}, & \{\psi^m, \psi^\ell\} \end{pmatrix}, \tag{23}$$

around the trivial and non-trivial fixed points. The unphysical degrees of freedom are eliminated by solving the second class constraints and using the associated Dirac brackets. We can also write the Hamiltonian in the reduced phase space.

This procedure is analogous to the construction of the physical phase space of gauge theories. In this case one introduces gauge fixing constraints in such a way to convert the first class constraints, associated to gauge invariances, to second class constraints. The gauge degrees of freedom are eliminated by solving the second class constraints.

When a non-local theory under investigation contains an infinite number of second class constraints, the dimension of the physical phase space might be reduced to be finite dimensional. This implies that with a suitable choice of variables the system would be described in terms of a local theory.

An alternative way to see the physical degrees of freedom is to compute the symplectic form, (9) or (18), on the surface Σ^* defined by the constraints (20) and (21).

3 Non-local harmonic oscillator

In this section we will apply the above formalism to an harmonic oscillator with a non-local interaction to illustrate how the reduced phase spaces come out within this frameworks. We will compute the perturbative and the non-perturbative phase space for this model around the unique fixed point $q = 0$. The dimension of the reduced phase space is finite dimensional in agreement with [13],[15].

The Lagrangian of the non-local harmonic oscillator is given by

$$L^{non}(q(t)) = \frac{1}{2}\dot{q}(t)^2 - \frac{\omega^2}{2}q(t)^2 + \frac{g}{4}\int dt' q(t) e^{-|t-t'|} q(t'). \tag{24}$$

The associated 1+1 dimensional Lagrangian density $\mathcal{L}(t, \lambda)$ is given by

$$\mathcal{L}(t, \lambda) = \frac{1}{2}Q'(t, \lambda)^2 - \frac{\omega^2}{2}Q(t, \lambda)^2 + \frac{g}{4}\int d\lambda' Q(t, \lambda) e^{-|\lambda-\lambda'|} Q(t, \lambda') \tag{25}$$

and the Hamiltonian (7) is

$$H = \sum_{m=0}^{\infty} p_m q^{m+1} - [\frac{1}{2}(q^1)^2 - \frac{\omega^2}{2}(q^0)^2 + \frac{g}{2} q^0 \sum_{r=0} q^{2r}]. \tag{26}$$

The momentum constraint (10) in the basis (20) implies the constraints

$$\begin{aligned} \varphi_{2m} &= p_{2m} - q^1 \delta_{0,m} + \frac{g}{2}\sum_{r=0}^{\infty} q^{2r+1} \approx 0, \\ \varphi_{2m+1} &= p_{2m+1} - \frac{g}{2}\sum_{r=0}^{\infty} q^{2r} \approx 0, \qquad (m \geq 0). \end{aligned} \tag{27}$$

The EL constraint (12) is

$$\psi(t,\lambda) = Q''(t,\lambda) + \omega^2 Q(t,\lambda) - \frac{g}{2}\int d\lambda' \, e^{-|\lambda-\lambda'|} Q(t,\lambda') \approx 0, \tag{28}$$

in the basis (21) it implies

$$\psi^m = q^{m+2} + \omega^2 q^m - g\sum_{r=0}^{\infty} q^{m+2r} \approx 0, \qquad (m \geq 0). \tag{29}$$

Note $\frac{d}{dt}\psi^m = \psi^{m+1}$, i.e the constraints ψ^m preserve their form in time. The EL constraint (28) with the Hamilton's equation, (6), $\dot{Q} = Q'$, i.e., $q(t+\lambda) \equiv Q(t,\lambda)$ reproduces the EL equation of the non-local action (24),

$$\ddot{q}(t) + \omega^2 q(t) - \frac{g}{2}\int dt' \, e^{-|t-t'|} q(t') = 0. \tag{30}$$

3.1 Simple Harmonic Oscillator

When the coupling constant g vanishes this system is an ordinary harmonic oscillator. The constraints (27) and (29) are

$$\varphi_0 = p_0 - q^1 \approx 0, \qquad \varphi_m = p_m \approx 0, \ (m \geq 1), \tag{31}$$

$$\psi^m = q^{m+2} + \omega^2 q^m \approx 0, \qquad (m \geq 0). \tag{32}$$

They are second class constraints and are paired as

$$\varphi_0 = p_0 - q^1, \quad \text{with} \quad \varphi_1 = p_1 \tag{33}$$

to eliminate the canonical pair (q^1, p_1). Analogously,

$$\varphi_{m+2} = p_{m+2}, \quad \text{and} \quad \psi^m = q^{m+2} + \omega^2 q^m \tag{34}$$

are paired to eliminate the canonical pairs (q^{m+2}, p_{m+2}) for $m+2 \geq 2$. As a result, all canonical pairs (q^m, p_m) with $m \geq 1$ are expressed in terms of the canonical pair (q^0, p_0). The Dirac bracket in the reduced phase space of (q^0, p_0) coincides with the

Poisson bracket, since the sets of the second class constraints (33) and (34) have the standard forms[3]. The Hamiltonian H^* in the reduced space Σ^* becomes the one of an ordinary harmonic oscillator

$$H^* \;=\; \frac{1}{2}(p_0)^2 \;+\; \frac{\omega^2}{2}(q^0)^2. \tag{35}$$

The dimension of the reduced phase space is 2 (one canonical pair) as was expected.

3.2 Perturbative Pairing

When the coupling constant g is small and the perturbative treatment is allowed the physical degrees of freedom of the system are same as in the $g=0$ case. This is because all constraints remain in the second class and we can make the same pairing of constraints as in the free case. To show it explicitly we should rewrite the constraints in the standard form.

The EL constraints (29) are expressed using by themselves iteratively as

$$\begin{aligned} \widetilde{\psi}^{2\ell} &= q^{2\ell+2} + (-1)^\ell k^{2\ell+2} q^0 \;\approx\; 0, \\ \widetilde{\psi}^{2\ell+1} &= q^{2\ell+3} + (-1)^\ell k^{2\ell+2} q^1 \;\approx\; 0, \end{aligned} \tag{36}$$

where

$$k^2 \;=\; \omega^2 - \frac{g}{(\omega^2+1)} + \frac{g^2}{(\omega^2+1)^3} - \frac{2\;g^3}{(\omega^2+1)^5} + \ldots \;=\; \frac{(\omega^2-1)+\sqrt{(\omega^2+1)^2-4g}}{2}. \tag{37}$$

The two k's given from (37) are solutions of the characteristic equation associated with the EL equation, (30),

$$k^2 - \omega^2 + g\,\frac{1}{1+k^2} \;=\; 0. \tag{38}$$

The other two solutions of (38) can not be obtained by iteration in g. (See next subsection.) Instead of working with the momentum constraints (27) it is more convenient to introduce the following combinations

$$\begin{aligned} \widetilde{\varphi}_{2\ell} &= \varphi_{2\ell} - \frac{g}{2}\sum_{r=0}^{\infty} \widetilde{\psi}^{2r+1} \;=\; p_{2\ell} - q^1\;\delta_{\ell,0} + \frac{g}{2}\frac{q^1}{1+k^2} \;\approx\; 0, \\ \widetilde{\varphi}_{2\ell+1} &= \varphi_{2\ell+1} + \frac{g}{2}\sum_{r=0}^{\infty} \widetilde{\psi}^{2r} \;=\; p_{2\ell+1} - \frac{g}{2}\frac{q^0}{1+k^2} \;\approx\; 0. \end{aligned} \tag{39}$$

[3]When a set of second class constraints has the form $\{p=0, q=f(\mathbf{q},\mathbf{p})\}$ the canonical pair (p,q) is expressed in terms of the remaining variables $(\mathbf{p},\mathbf{q})$. The Dirac bracket is defined by

$$\{A,B\}^* = \{A,B\} - \{A,p\}\{q-f(\mathbf{q},\mathbf{p}),B\} + \{A,q-f(\mathbf{q},\mathbf{p})\}\{p,B\}.$$

The values of Dirac bracket of $(\mathbf{q},\mathbf{p})$ are same as those of the Poisson bracket. We refer them as a standard form of second class constraints. $\{q=0, p=g(\mathbf{q},\mathbf{p})\}$ is also the standard form of second class constraints.

In order to rewrite the constraints in the standard form we perform a canonical transformation generated by

$$W = \sum_{m=0}^{\infty}\left(q^{2m}(\widetilde{p}_{2m} - \frac{g}{2}\frac{q^1}{1+k^2}) + q^{2m+1}(\widetilde{p}_{2m+1} + \frac{g}{2}\frac{q^0}{1+k^2})\right) + \frac{g}{2}\frac{q^0q^1}{(1+k^2)^2} \tag{40}$$

which gives

$$\begin{aligned}
\widetilde{q}^{\ell} &= q^{\ell}, \\
p_0 &= \widetilde{p}_0 - \frac{g}{2}\frac{q^1}{1+k^2} + g\frac{q^1}{(1+k^2)^2} + \frac{g}{2(1+k^2)}\sum_{r=0}^{\infty} \widetilde{\psi}^{2r+1}, \\
p_1 &= \widetilde{p}_1 + \frac{g}{2}\frac{q^0}{1+k^2} - \frac{g}{2(1+k^2)}\sum_{r=0}^{\infty} \widetilde{\psi}^{2r}, \\
p_{2m} &= \widetilde{p}_{2m} - \frac{g}{2}\frac{q^1}{1+k^2}, \qquad p_{2m+1} = \widetilde{p}_{2m+1} + \frac{g}{2}\frac{q^0}{1+k^2}, \quad (m \geq 1).
\end{aligned} \tag{41}$$

In terms of the new canonical variables we have

$$\begin{aligned}
\widetilde{\varphi}_0 &= \widetilde{p}_0 - (1 - \frac{g}{(1+k^2)^2})\widetilde{q}^1 \approx 0, \qquad \widetilde{\varphi}_1 = \widetilde{p}_1 \approx 0, \\
\widetilde{\varphi}_{2\ell+2} &= \widetilde{p}_{2\ell+2} \approx 0, \qquad \widetilde{\psi}^{2\ell} = q^{2\ell+2} + (-1)^{\ell}k^{2\ell+2}\widetilde{q}^0 \approx 0, \\
\widetilde{\varphi}_{2\ell+3} &= \widetilde{p}_{2\ell+3} \approx 0, \qquad \widetilde{\psi}^{2\ell+1} = q^{2\ell+3} + (-1)^{\ell}k^{2\ell+2}\widetilde{q}^1 \approx 0, \quad (\ell \geq 0).
\end{aligned} \tag{42}$$

The constraints (42) are used to eliminate canonical pairs $(\widetilde{q}^m, \widetilde{p}_m)$, $(m \geq 1)$ in terms of $(\widetilde{q}^0, \widetilde{p}_0)$. The Dirac bracket between $(\widetilde{q}^0, \widetilde{p}_0)$ is $\{\widetilde{q}^0, \widetilde{p}_0\}^* = 1$. Thus the reduced phase space has dimension 2 and the Hamiltonian is

$$H = \frac{1}{2}\left(\frac{(1+k^2)^2}{(1+k^2)^2 - g}\,(\widetilde{p}_0)^2 + \frac{(1+k^2)^2\omega^2 - g(1+2k^2)}{(1+k^2)^2}(\widetilde{q}^0)^2\right). \tag{43}$$

In the reduced space we have an ordinary harmonic oscillator with the frequency k,

$$\frac{(1+k^2)^2}{(1+k^2)^2 - g}\,\frac{(1+k^2)^2\omega^2 - g(1+2k^2)}{(1+k^2)^2} = k^2, \tag{44}$$

and mass

$$M = \frac{(1+k^2)^2 - g}{(1+k^2)^2}. \tag{45}$$

3.3 Non Perturbative Pairing

If the coupling constant is not small there is another possible pairing for the constraints (27) and (29). Let us consider a canonical transformation generated by

$$W = \widetilde{p}_0 q^0 + \widetilde{p}_1(\sum_{j=0} q^{2j+1}) + \widetilde{p}_2(\sum_{j=0} q^{2j+2}) + \sum_{j=3} \widetilde{p}_j q^j - \frac{g}{2}(\sum_{j=0} q^{2j+1})(\sum_{\ell=0} q^{2\ell+2}). \tag{46}$$

It makes $(\sum_{j=1,odd} q^j)$ and $(\sum_{j=2,even} q^j)$ to be new coordinates,

$$\begin{array}{rclrcll}
\widetilde{q}^0 &=& q^0, & p_0 &=& \widetilde{p}_0 & \\
\widetilde{q}^1 &=& (\sum_{j=1,odd} q^j), & p_1 &=& \widetilde{p}_1 - \frac{g}{2}\widetilde{q}^2 & \\
\widetilde{q}^2 &=& (\sum_{j=2,even} q^j), & p_2 &=& \widetilde{p}_2 - \frac{g}{2}\widetilde{q}^1 & \\
\widetilde{q}^{2\ell+1} &=& q^{2\ell+1}, & p_{2\ell+1} &=& \widetilde{p}_{2\ell+1} + \widetilde{p}_1 - \frac{g}{2}\widetilde{q}^2, & (\ell \geq 1) \\
\widetilde{q}^{2\ell} &=& q^{2\ell}, & p_{2\ell} &=& \widetilde{p}_{2\ell} + \widetilde{p}_2 - \frac{g}{2}\widetilde{q}^1, & (\ell \geq 2).
\end{array} \tag{47}$$

In terms of new variables the constraints (27) and (29) are

$$\begin{aligned}
\varphi_0 &= \widetilde{p}_0 - (\widetilde{q}^1 - \sum_{j=1} \widetilde{q}^{2j+1}) + \frac{g}{2}\widetilde{q}^1, \\
\varphi_1 &= \widetilde{p}_1 - \frac{g}{2}\widetilde{q}^0 - g\widetilde{q}^2, \\
\varphi_\ell &= \widetilde{p}_\ell, \qquad (\ell \geq 2).
\end{aligned} \tag{48}$$

$$\begin{aligned}
\psi^0 &= (1-g)\widetilde{q}^2 + (\omega^2 - g)\widetilde{q}^0 - \sum_{j=2} \widetilde{q}^{2j}, \\
\psi^1 &= (1-\omega^2)\widetilde{q}^3 + (\omega^2 - g)\widetilde{q}^1 - \omega^2 \sum_{j=2} \widetilde{q}^{2j+1}, \\
\psi^2 &= (1-\omega^2)\widetilde{q}^4 + (\omega^2 - g)\widetilde{q}^2 - \omega^2 \sum_{j=2} \widetilde{q}^{2j+2}, \\
\psi^\ell &= (1-g)\widetilde{q}^{\ell+2} + (\omega^2 - g)\widetilde{q}^\ell - g \sum_{r=2} \widetilde{q}^{\ell+2r}, \qquad (\ell \geq 3).
\end{aligned} \tag{49}$$

The Hamiltonian (26) becomes

$$\begin{aligned}
H &= \widetilde{p}_0(\widetilde{q}^1 - \sum_{r=1} \widetilde{q}^{2r+1}) + (\widetilde{p}_1 - \frac{g}{2}\widetilde{q}^2)\widetilde{q}^2 + (\widetilde{p}_2 - \frac{g}{2}\widetilde{q}^1) \sum_{r=1} \widetilde{q}^{2r+1} + \sum_{r=3,odd} \widetilde{p}_r \widetilde{q}^{r+1} \\
&+ \sum_{r=4,even} \widetilde{p}_r \widetilde{q}^{r+1} - [\frac{1}{2}(\widetilde{q}^1 - \sum_{r=1} \widetilde{q}^{2r+1})^2 - \frac{\omega^2}{2}(\widetilde{q}^0)^2 + \frac{g}{2}\, \widetilde{q}^0\, (\widetilde{q}^0 + \widetilde{q}^2)].
\end{aligned} \tag{50}$$

There are two inequivalent pairings:

$$i), \qquad (\varphi_0, \varphi_1), \quad (\varphi_{\ell+2}, \psi^\ell), \quad (\ell \geq 0), \tag{51}$$

it is the perturbative pairing discussed in the previous subsection.

$$ii), \qquad (\varphi_0, \varphi_3), \quad (\varphi_1, \varphi_2), \quad (\varphi_{\ell+4}, \psi^\ell), \quad (\ell \geq 0). \tag{52}$$

It is a non-perturbative paring because it requires an inverse power of g. φ_1 and φ_2 are paired to eliminate $(\widetilde{q}^2, \widetilde{p}_2)$ as

$$\widetilde{p}_2 = 0, \qquad \widetilde{q}^2 = \frac{1}{g}\,\widetilde{p}_1 - \frac{1}{2}\widetilde{q}^0. \tag{53}$$

φ_0 and φ_3 are paired to eliminate $(\widetilde{q}^3, \widetilde{p}_3)$ as

$$\widetilde{p}_3 = 0, \qquad \sum_{r=1} \widetilde{q}^{2r+1} = -\widetilde{p}_0 + \widetilde{q}^1 - \frac{g}{2}\widetilde{q}^1. \tag{54}$$

Other constraints, $(\varphi_{\ell+4}, \psi^{\ell})$, $(\ell \geq 0)$, are used to solve $(\widetilde{q}^j, \widetilde{p}_j), (j \geq 4)$. Thus the reduced phase space is spanned by $(\widetilde{q}^j, \widetilde{p}_j), (j = 0, 1)$ and is 4 dimensional.

The Hamiltonian in the reduced space is

$$H = \frac{1}{2}(\widetilde{p}_0 + \frac{g}{2}\widetilde{q}^1)^2 + \frac{g}{2}(\frac{1}{g}\widetilde{p}_1 - \frac{1}{2}\widetilde{q}^0)^2 - \frac{g}{2}(\widetilde{q}^1)^2 + \frac{\omega^2}{2}(\widetilde{q}^0)^2 - \frac{g}{2}(\widetilde{q}^0)^2. \tag{55}$$

The non-vanishing Dirac brackets for $(\widetilde{q}^0, \widetilde{q}^1, \widetilde{p}_0, \widetilde{p}_1)$ are given by $\{\widetilde{q}^0, \widetilde{p}_0\}^* = \{\widetilde{q}^1, \widetilde{p}_1\}^* = 1$. Further canonical transformation shows it is a system of two harmonic oscillators

$$\begin{aligned} H &= \frac{1}{2}(\widehat{p}_0)^2 + \frac{k_+^2}{2}(\widehat{q}^0)^2 + \frac{1}{2}(\widehat{p}_1)^2 + \frac{k_-^2}{2}(\widehat{q}^1)^2, \\ k_\pm^2 &= \frac{(\omega^2 - 1) \pm \sqrt{(\omega^2+1)^2 - 4g}}{2}. \end{aligned} \tag{56}$$

The frequencies $k_\pm^2$ are solutions of (38). For small value of g, $k_+^2 > 0$ represents the ordinary harmonic oscillator mode while $k_-^2 < 0$ mode has potential of inverse sign.

Summing up, the reduced perturbative phase space around $q = 0$ is two dimensional, which is the same dimension as the free theory ($g = 0$). Instead the non-perturbative one is four dimensional. Notice that in both cases there is an infinite dimensional reduction of the original degrees of freedom.

4 P-adic particle

P-adic string theory [12] has been used as a toy model to study tachyon condensation. For spatially homogeneous configurations (p-adic particle) the rolling of the tachyon has been analyzed by constructing a classical time dependent solution . This solution has oscillations in time with ever-growing amplitude $\phi(t) = \sum_n a_n e^{nt}$ [4]. Here we will analyze the reduced phase space of the p-adic particle.

The action for the p-adic particle is

$$S = \frac{1}{g_p^2} \int dt \, [-\frac{1}{2} \, q(t) \, p^{\frac{1}{2}\partial_t^2} \, q(t) + \frac{1}{p+1} \, q(t)^{p+1} \,], \tag{57}$$

where $\frac{1}{g_p} = \frac{1}{g^2}\frac{p^2}{p-1}$ and p is a prime number. The Euler-Lagrange equation is

$$p^{\frac{\partial_t^2}{2}} \, q(t) = q(t)^p. \tag{58}$$

In the following we examine the $p = 2$ system. After rescaling the Lagrangian density, $\mathcal{L}(t, \lambda)$ can be defined by

$$\mathcal{L}(t, \sigma) = -\frac{1}{2} \, Q(t, \sigma) \, e^{\partial_\sigma^2} Q(t, \sigma) + \frac{g}{3} \, Q(t, \sigma)^3. \tag{59}$$

The EL constraint (12) is

$$\psi(t, \lambda, [Q]) \equiv \int d\sigma \, \mathcal{E}(t; \sigma, \lambda) = - e^{\partial_\lambda^2} \, Q(t, \lambda) + g \, Q(t, \lambda)^2 \approx 0 \tag{60}$$

and the momentum constraint (10) becomes

$$\begin{aligned}\varphi(t,\lambda,[Q,P]) &\equiv P(t,\lambda) - \int d\sigma\, \chi(\lambda,-\sigma)\, \mathcal{E}(t;\sigma,\lambda) \\ &= P(t,\lambda) \;-\frac{1}{2}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+m+1)!}[(\partial_\sigma^{2\ell+1}Q(t,\sigma))|_{\sigma=0}\ (-\partial_\lambda)^{2m}\delta(\lambda) \\ &\qquad -\ (\partial_\sigma^{2\ell}Q(t,\sigma))|_{\sigma=0}\ (-\partial_\lambda)^{2m+1}\delta(\lambda)] \approx 0, \end{aligned} \tag{61}$$

where we have used,

$$\chi(\lambda,-\sigma)\partial_\sigma^\ell\delta(\lambda-\sigma) \;=\; \sum_{m=0}^{\ell-1}(-)^m\partial_\sigma^{\ell-1-m}\delta(\sigma)\ \partial_\lambda^m\delta(\lambda), \qquad (\text{for } \ \ell\geq 1). \tag{62}$$

The Hamiltonian of this system is

$$\begin{aligned} H &= \int d\lambda\ P(t,\lambda)Q'(t,\lambda)\ -\ \mathcal{L}(t,0) \\ &= \int d\lambda\ P(t,\lambda)Q'(t,\lambda)\ +\ \frac{1}{2}\ Q(t,0)\ (e^{\partial_\sigma^2}Q(t,\sigma))|_{\sigma=0}\ -\ \frac{g}{3}\ Q(t,0)^3. \end{aligned} \tag{63}$$

In the basis $(e_n(\lambda), e^n(\lambda))$, (15), the constraints and the Hamiltonian are

$$\psi^m \;=\; -\sum_{\ell=0}^{\infty}\frac{q^{m+2\ell}}{\ell!}\ +\ g\sum_{\ell=0}^{m}{}_mC_\ell\ q^{m-\ell}q^\ell\ \approx\ 0, \tag{64}$$

$$\begin{aligned}\varphi_{2m} &= p_{2m} - \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+m+1)!}\ q^{2\ell+1}\ \approx\ 0, \\ \varphi_{2m+1} &= p_{2m+1} + \frac{1}{2}\sum_{\ell=0}^{\infty}\frac{1}{(\ell+m+1)!}\ q^{2\ell}\ \approx\ 0, \end{aligned} \tag{65}$$

$$H \;=\; \sum_{m=0}^{\infty}p_m q^{m+1}\ +\ \frac{1}{2}\ q^0\ \sum_{m=0}^{\infty}\frac{q^{2m}}{m!}\ -\ \frac{g}{3}(q^0)^3. \tag{66}$$

where ${}_mC_\ell = \frac{m!}{(m-\ell)!\ell!}$ is the combinatorial coefficient. Note that ψ^m preserves its form in time.

4.1 Free p-adic particle

We first consider the case of free action. When $g=0$ the EL constraints (64) can be recombined as

$$\widetilde{\psi}^m \;\equiv\; -\sum_{r=0}^{\infty}\frac{(-)^r}{r!}\ \psi^{2r+m} \;=\; q^m\ \approx\ 0. \tag{67}$$

The momentum constraints (65) are also simplified by making combinations with the EL constraints

$$\begin{aligned}\widetilde{\varphi}_{2m} &\equiv \varphi_{2m}\ -\ \frac{1}{2}\sum_{r=0}^{\infty}\frac{(-)^r\ \psi^{2r+1}}{m!r!(m+r+1)} \;=\; p_{2m}\ \approx\ 0, \\ \widetilde{\varphi}_{2m+1} &\equiv \varphi_{2m+1}\ +\ \frac{1}{2}\sum_{r=0}^{\infty}\frac{(-)^r\ \psi^{2r}}{m!r!(m+r+1)} \;=\; p_{2m+1}\ \approx\ 0. \end{aligned} \tag{68}$$

The constraints $(\widetilde{\varphi}_m, \widetilde{\psi}^m) = (p_m, q^m)$ are second class. All canonical variables (q^m, p_m) vanish in the reduced phase space and therefore the reduced phase space is 0-dimensional.

An analogous conclusion was obtained for a solution in the case of the purely cubic bosonic string field theory in [20]. The Hamiltonian of this system (66) vanishes in the physical phase space.

4.2 Perturbative Pairing

We consider the reduced phase space around the fixed point $q = 0$. If we make the same combinations of constraints as (67) and (68)

$$\begin{aligned}
\widetilde{\psi}^m &\equiv - \sum_{r=0}^{\infty} \frac{(-)^r}{r!} \psi^{2r+m} \\
&= q^m - g \sum_{r=0}^{\infty} \frac{(-)^r}{r!} \sum_{\ell=0}^{2r+m} {}_{2r+m}C_\ell \, q^{2r+m-\ell} q^\ell \approx 0, \qquad (69)\\
\widetilde{\varphi}_{2m} &\equiv \varphi_{2m} - \frac{1}{2} \sum_{s=0}^{\infty} \left(\frac{(-)^s}{(m+s+1)m!s!}\right) \psi^{2s+1} \\
&= p_{2m} - \frac{g}{2} \sum_{s=0}^{\infty} \left(\frac{(-)^s}{(m+s+1)m!s!}\right) \sum_{\ell=0}^{2s+1} {}_{2s+1}C_\ell \, q^{2s+1-\ell} q^\ell \approx 0, \\
\widetilde{\varphi}_{2m+1} &\equiv \varphi_{2m+1} + \frac{1}{2} \sum_{s=0}^{\infty} \left(\frac{(-)^s}{(m+s+1)m!s!}\right) \psi^{2s} \\
&= p_{2m+1} + \frac{g}{2} \sum_{s=0}^{\infty} \left(\frac{(-)^s}{(m+s+1)m!s!}\right) \sum_{\ell=0}^{2s} {}_{2s}C_\ell \, q^{2s-\ell} q^\ell \approx 0. \qquad (70)
\end{aligned}$$

The only possible forms of the constraints obtained by iterations are

$$\widehat{\psi}^m = q^m \approx 0, \qquad \widehat{\varphi}^m = p_m \approx 0. \qquad (71)$$

Therefore the system has zero degrees of freedom and zero Hamiltonian as in the free p-adic particle. Obviously the symplectic two form (18) vanishes. It means that at the local minimum $q = 0$ there is no non-trivial solution. That is there is no excitation modes at the local minimum (tachyonic vacuum). Following Sen we may interpret this result saying that there are not closed string states at the minimum [21].

4.3 Physical space around the fixed point $q = \frac{1}{g}$

The EL constraints (69) have non-perturbative fixed point as well. In order to see it we return to (60) and note that it has a constant solution $q = \frac{1}{g}$ in addition to the $q = 0$ solution. We introduce new canonical variables as

$$\begin{aligned}
q^0 &= \frac{1}{g} + \widetilde{q}^0, \qquad q^j = \widetilde{q}^j, \quad (j > 0) \\
p^{2m} &= \widetilde{p}^{2m}, \qquad p_{2m+1} = \widetilde{p}_{2m+1} - \frac{1}{2g(m+1)!}. \qquad (72)
\end{aligned}$$

In terms of new variables the constraints are

$$\widetilde{\psi}^m \;=\; 2\widetilde{q}^m \;-\; \sum_{\ell=0}^{\infty} \frac{\widetilde{q}^{m+2\ell}}{\ell!} \;+\; g \sum_{\ell=0}^{m} {}_mC_\ell \; \widetilde{q}^{m-\ell}\widetilde{q}^{\ell} \;\approx\; 0, \tag{73}$$

$$\widetilde{\varphi}_{2m} \;=\; \widetilde{p}_{2m} - \frac{1}{2}\sum_{\ell=0}^{\infty} \frac{1}{(\ell+m+1)!}\; \widetilde{q}^{2\ell+1} \;\approx\; 0,$$

$$\widetilde{\varphi}_{2m+1} \;=\; \widetilde{p}_{2m+1} + \frac{1}{2}\sum_{\ell=0}^{\infty} \frac{1}{(\ell+m+1)!}\; \widetilde{q}^{2\ell} \;\approx\; 0 \tag{74}$$

Note that $\frac{d}{dt}\widetilde{\psi}^m = \widetilde{\psi}^{m+1}$. The Hamiltonian (66) is

$$H \;=\; \sum_{m=0}^{\infty} \widetilde{p}_m \widetilde{q}^{m+1} + \frac{1}{6g^2} - (\widetilde{q}^0)^2 - \frac{g}{3}(\widetilde{q}^0)^3 + \frac{1}{2}\,\widetilde{q}^0 \sum_{m=0}^{\infty} \frac{\widetilde{q}^{2m}}{m!}. \tag{75}$$

As we have separated the g^{-1} term, $\widetilde{q}^0$ and $\widetilde{q}^j$'s are expected to be regular in $g \to 0$ and they can be determined iteratively. In matrix form the EL constraints (73) are

$$\begin{pmatrix} \widetilde{\psi}^0 \\ \widetilde{\psi}^2 \\ \widetilde{\psi}^4 \\ \widetilde{\psi}^6 \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & -\frac{1}{4!} & \dots & \\ 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & -\frac{1}{4!} & \dots \\ 0 & 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & \dots \\ 0 & 0 & 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & \dots \\ \dots\dots\dots & & & & & & \end{pmatrix} \begin{pmatrix} \widetilde{q}^0 \\ \widetilde{q}^2 \\ \widetilde{q}^4 \\ \widetilde{q}^6 \\ \cdots \end{pmatrix} + g \begin{pmatrix} \widetilde{F}^0 \\ \widetilde{F}^2 \\ \widetilde{F}^4 \\ \widetilde{F}^6 \\ \cdots \end{pmatrix} \approx 0, \tag{76}$$

$$\begin{pmatrix} \widetilde{\psi}^1 \\ \widetilde{\psi}^3 \\ \widetilde{\psi}^5 \\ \widetilde{\psi}^7 \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & -\frac{1}{4!} & \dots & \\ 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & -\frac{1}{4!} & \dots \\ 0 & 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & -\frac{1}{3!} & \dots \\ 0 & 0 & 0 & 1 & -\frac{1}{1!} & -\frac{1}{2!} & \dots \\ \dots\dots\dots & & & & & & \end{pmatrix} \begin{pmatrix} \widetilde{q}^1 \\ \widetilde{q}^3 \\ \widetilde{q}^5 \\ \widetilde{q}^7 \\ \cdots \end{pmatrix} + g \begin{pmatrix} \widetilde{F}^1 \\ \widetilde{F}^3 \\ \widetilde{F}^5 \\ \widetilde{F}^7 \\ \cdots \end{pmatrix} \approx 0, \tag{77}$$

where

$$\widetilde{F}^l = \sum_{m=0}^{l} {}_lC_m \; \widetilde{q}^{l-m}\widetilde{q}^m \tag{78}$$

In this case to find the pairing among the ψ constraints and φ constraints is not obvious. Here we are going to use another way that consists in solving the ψ constraints [4].

Lets us indicate by $\mathcal{D}$ the matrix appearing in the equation (76) and (77). The matrix $\mathcal{D}$ has unit determinant[5] and has a "formal inverse",

$$\mathcal{D}^{-1} \;=\; \begin{pmatrix} 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots & \\ 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots \\ 0 & 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \dots \\ 0 & 0 & 0 & 1 & \frac{1}{1!} & \dots \\ \dots\dots\dots & & & & & \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots & \\ 0 & c_0 & c_1 & c_2 & c_3 & \dots \\ 0 & 0 & c_0 & c_1 & c_2 & \dots \\ 0 & 0 & 0 & c_0 & c_1 & \dots \\ \dots\dots\dots & & & & & \end{pmatrix} \tag{79}$$

[4]To see the procedure in the simple case of the harmonic oscillator see the appendix A

[5]We have assumed a regularization of $\mathcal{D} = \lim_{n\to\infty} \mathcal{D}_n$.

where c_n's are generated by

$$\frac{1}{2-e^{x}} \;=\; \sum_{n=0} c_n\, x^n \;=\; 1+x+\frac{3}{2}x^2+\frac{13}{6}x^3+.... \tag{80}$$

The infinite matrix $\mathcal{D}$ has an infinite number of eigen-vectors. In fact we have

$$\mathcal{D}\begin{pmatrix} 1 \\ \kappa^2 \\ \kappa^4 \\ \kappa^6 \\ \kappa^8 \\ \dots \end{pmatrix} \;=\; (2-e^{\kappa^2})\begin{pmatrix} 1 \\ \kappa^2 \\ \kappa^4 \\ \kappa^6 \\ \kappa^8 \\ \dots \end{pmatrix}, \tag{81}$$

and the eigen-value can vanish when κ satisfies

$$2 \;-\; e^{\kappa^2} \;=\; 0. \tag{82}$$

This equation has an infinite numbers of solutions

$$\kappa_r^2 \;=\; \log 2 + 2\pi i r, \qquad r\in Z. \tag{83}$$

Using the null vectors (81) with (83) we can rewrite the constraints (76) in the following equivalent forms

$$\begin{pmatrix} \widetilde{q}^0 \\ \widetilde{q}^2 \\ \widetilde{q}^4 \\ \widetilde{q}^6 \\ \dots \end{pmatrix} \;\approx\; \sum_r \begin{pmatrix} 1 \\ \kappa_r^2 \\ \kappa_r^4 \\ \kappa_r^6 \\ \dots \end{pmatrix} A^r(t) \;-\; g \begin{pmatrix} 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots & \\ 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots \\ 0 & 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \dots \\ 0 & 0 & 0 & 1 & \frac{1}{1!} & \dots \\ \dots\dots\dots & & & & & \end{pmatrix}\begin{pmatrix} \widetilde{F}^0 \\ \widetilde{F}^2 \\ \widetilde{F}^4 \\ \widetilde{F}^6 \\ \dots \end{pmatrix}, \tag{84}$$

$$\begin{pmatrix} \widetilde{q}^1 \\ \widetilde{q}^3 \\ \widetilde{q}^5 \\ \widetilde{q}^7 \\ \dots \end{pmatrix} \;\approx\; \sum_r \begin{pmatrix} \kappa_r^1 \\ \kappa_r^3 \\ \kappa_r^5 \\ \kappa_r^7 \\ \dots \end{pmatrix} A^r(t) \;-\; g \begin{pmatrix} 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots & \\ 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \frac{13}{6} & \dots \\ 0 & 0 & 1 & \frac{1}{1!} & \frac{3}{2!} & \dots \\ 0 & 0 & 0 & 1 & \frac{1}{1!} & \dots \\ \dots\dots\dots & & & & & \end{pmatrix}\begin{pmatrix} \widetilde{F}^1 \\ \widetilde{F}^3 \\ \widetilde{F}^5 \\ \widetilde{F}^7 \\ \dots \end{pmatrix} \tag{85}$$

where the functions $A^r(t)$, appearing as coefficients of the null vectors, should satisfy

$$\dot{A}^r(t) = \kappa_r\; A^r(t) \tag{86}$$

in order the constraints preserve their forms, (84) and (85), in time.

By iterations in g, $\widetilde{q}^j$ can be obtained in terms of the $A^r(t)$[6],

$$\begin{aligned} \widetilde{q}^m \;\approx\; & \sum_j A^j(\kappa_j)^m \;+\; g\sum_{jk}\frac{A^jA^k}{(e^{(\kappa_j+\kappa_k)^2}-2)}(\kappa_j+\kappa_k)^m \;+ \\ & +\; g^2\sum_{jk\ell}\frac{2A^\ell A^j A^k}{(e^{(\kappa_j+\kappa_k+\kappa_\ell)^2}-2)(e^{(\kappa_j+\kappa_k)^2}-2)}(\kappa_\ell+\kappa_j+\kappa_k)^m \;+\; .... \\ \equiv\; & f^m[A(t)]. \end{aligned} \tag{87}$$

[6]Special care must be paid in g^2 term when two κ's cancel in $\kappa_j+\kappa_k+\kappa_\ell$ since the denominator $(e^{(\kappa_j+\kappa_k+\kappa_\ell)^2}-2)$ vanishes in those cases.

It is the solution of EL constraints (73) and gives the solutions of the EL equation (58). The Moeller-Zwiebach solution corresponds to $A^0 = \text{real}, \kappa_0 = \sqrt{\log 2}$ and other $A^r = 0$ [4],

$$\begin{aligned} q(t) &= \frac{1}{g} + A^0 e^{\kappa_0 t} + g\frac{(A^0)^2}{(e^{(2\kappa_0)^2} - 2)} e^{(2\kappa_0)t} + \\ &+ g^2 \frac{2(A^0)^3}{(e^{(3\kappa_0)^2} - 2)(e^{(2\kappa_0)^2} - 2)} e^{(3\kappa_0)t} + \cdots \\ &= \frac{1}{g} + A^0 e^{\kappa_0 t} + g\frac{(A^0)^2}{14} e^{(2\kappa_0)t} + g^2 \frac{2(A^0)^3}{51014} e^{(3\kappa_0)t} + \cdots . \end{aligned} \tag{88}$$

Solutions with infinite number of parameters have also been found by Schnabl, Sen and Zwiebach [19][7]. A solution with A^1, $A^{1*} \neq 0$ is given by

$$\begin{aligned} q(t) &= \frac{1}{g} + \left(A^1 e^{(b_1 + ic_1)t} + A^{1*} e^{(b_1 - ic_1)t}\right) + \\ &+ g\left(\frac{(A^1)^2}{(e^{4(b_1^2 - c_1^2)} - 2)} e^{2(b_1 + ic_1)t} + \frac{2A^1 A^{1*}}{(e^{4b_1^2} - 2)} e^{2b_1 t} + \frac{(A^{1*})^2}{(e^{4(b_1^2 - c_1^2)} - 2)} e^{2(b_1 - ic_1)t}\right) + \cdots , \end{aligned} \tag{89}$$

where $b_1^2 - c_1^2 = \log 2$, $b_1 c_1 = \pi N, (b_1 > 0)$ for some non-zero integer N. The sums are taken over $A^j = A^1, A^{1*}$ and $\kappa_1 = (b_1 + ic_1), \kappa_1^* = (b_1 - ic_1)$.

The momentum constraints (74) can be also solved using (87)

$$\begin{aligned} \widetilde{p}_{2m} - \frac{1}{2} \sum_{\ell=0} \frac{1}{(\ell + m + 1)!} \; f^{2\ell+1}[A] &\approx 0, \\ \widetilde{p}_{2m+1} + \frac{1}{2} \sum_{\ell=0} \frac{1}{(\ell + m + 1)!} \; f^{2\ell}[A] &\approx 0. \end{aligned} \tag{90}$$

Since we have the solutions of the coordinates q^m and the momenta p_m in terms of infinite arbitrary constants A^r we could compute the symplectic two form (18) in terms of these constants. At lowest order in g

$$\begin{aligned} \Omega &= \sum_m dq^m \wedge \; dp_m \\ &= \frac{1}{2} \sum_{rs} dA^r \wedge dA^s \; (\kappa_r - \kappa_s) \; [\sum_{m,\ell=0} \frac{1}{(m + \ell + 1)!} (\kappa_r)^{2\ell} (\kappa_s)^{2m}]. \end{aligned} \tag{91}$$

In the last sum terms with $\kappa_s^2 \neq \kappa_r^2$ are $\frac{e^{\kappa_r^2} - e^{\kappa_s^2}}{\kappa_r^2 - \kappa_s^2}$ and vanish using (82). On the other hand terms with $\kappa_s^2 = \kappa_r^2$ becomes $e^{\kappa_r^2}$. Then Ω becomes

$$\Omega = \sum_{rs} dA^r \wedge dA^s \; (\kappa_r - \kappa_s) \; |_{\kappa_s^2 = \kappa_r^2} = \sum_r \; 2\kappa_r \; dA^r \wedge dA^{-r} \tag{92}$$

where only $\kappa_s + \kappa_r = 0$ terms remain non-vanishing.

[7]We are grateful to Martin Schnabl for discussions.

Similarly if we use the same formula in the Hamiltonian it becomes , at lowest order in g ,

$$\begin{aligned} H &= \frac{1}{6g^2} - \frac{1}{2}\sum_{rs} A^r A^s\ \kappa_r(\kappa_r - \kappa_s)\ [\sum_{m,\ell=0} \frac{1}{(m+\ell+1)!}(\kappa_r)^{2\ell}(\kappa_s)^{2m}] \\ &= \frac{1}{6g^2} - 2\sum_r \kappa_r^2\ A^r A^{-r}. \end{aligned} \tag{93}$$

The forms of the symplectic two form Ω (92) and the Hamiltonian H (93) show that the p-adic particle has an infinite dimensional physical phase space around the fixed point $\frac{1}{g}$.

So far we are assuming $q(t)$ to be any sufficiently smooth functions. In the discussions of tachyon condensation we are interested in the field $q(t)$ which start from the local maximum $q(t) = \frac{1}{g}$ at $t = -\infty$. It restricts the modes only with

$$Re(\kappa_r)\ >\ 0. \tag{94}$$

The solutions (88) and (89) are satisfying this requirement. In restricting the function space by imposing (94) there appears no term satisfying $\kappa_s + \kappa_r = 0$ in the sum in (91). The symplectic two form Ω (92) vanishes identically. Similarly the Hamiltonian H (93) becomes a constant $\frac{1}{6g^2}$, the asymptotic value of energy at $t = -\infty$. (They are shown explicitly up to g^2.)

These results imply that the unstable and stable manifold of the non-perturbative fixed point $q = \frac{1}{g}$ is a infinite lagrangian submanifold of T^*J. The manifold of solutions relevant for the rolling tachyon, unstable submanifold, is infinite dimensional without a phase space structure. See fig. 1 for a summary of these results.

If we compute the energy (93) without imposing any boundary conditions, solutions do not have a definite sign of the energy as we will see in Appendix B. It corresponds to the general property of the non-local theories.

5 String Field theory

In order to study the rolling of tachyon we need to know the potential. The natural framework to compute it in string theory is the string field theory. The level truncation is an approximation scheme [22] to compute the potential. For the lowest level truncation approximation and for spatially homogeneous configurations the action is given by

$$S = \int dt(\frac{1}{2}\dot{q}(t)^2 + \frac{M^2}{2}q(t)^2 - \frac{g}{3}\tilde{q}(t)^3) \tag{95}$$

where

$$\tilde{q}(t) = e^{-\partial_t^2} q(t). \tag{96}$$

The Euler-Lagrange equation is

$$\ddot{q}(t) - M^2 q(t)\ +\ g\ e^{-\partial_t^2}(e^{-\partial_t^2} q(t))^2\ =\ 0. \tag{97}$$

It has been examined as a non-local system in [13].

In the present formalism the associated Lagrangian density $\mathcal{L}(t,\lambda)$ is

$$\mathcal{L}(t,\lambda) \;=\; \frac{1}{2}Q'(t,\lambda)^2 + \frac{M^2}{2}Q(t,\lambda)^2 - \frac{g}{3}\tilde{Q}(t,\lambda)^3 \qquad (98)$$

with

$$\tilde{Q}(t,\lambda) = e^{-\partial_\lambda^2}Q(t,\lambda).$$

The EL constraint (12) is

$$\psi(t,\lambda) \;=\; -\partial_\lambda^2 Q(t,\lambda) + M^2 Q(t,\lambda) \;-\; g\; e^{-\partial_\lambda^2}(\tilde{Q}(t,\lambda))^2 \;\approx\; 0, \qquad (99)$$

the momentum constraint (10) is

$$\begin{aligned}\varphi(t,\lambda) \;=\;& P(t,\lambda) - Q'(t,0)\delta(\lambda) + g\int d\sigma \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m+1}}{(n+m+1)!}\\ &\Big[\partial_\sigma^{2n+1}\delta(\sigma)(-\partial_\lambda)^{2m}\delta(\lambda) + \partial_\sigma^{2n}\delta(\sigma)(-\partial_\lambda)^{2m+1}\delta(\lambda)\Big](\tilde{Q}(t,\sigma))^2 \;\approx\; 0\end{aligned}$$

and the Hamiltonian (7) is

$$H = \int d\lambda P(t,\lambda)Q'(t,\lambda) - \frac{1}{2}Q'(t,0)^2 - \frac{M^2}{2}Q(t,0)^2 + \frac{g}{3}\widetilde{Q}(t,0)^3. \qquad (100)$$

In the basis (14) the constraints are given by

$$\begin{aligned}\varphi_{2m} \;&=\; p_{2m} \;-\; q^1\,\delta_{m,0} \;-\; 2\,g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} B^{eo}_{m,r,s}\, q^{2r}\, q^{2s+1},\\ \varphi_{2m+1} \;&=\; p_{2m+1} \;+\; g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(B^{ee}_{m,r,s}\, q^{2r}\, q^{2s} + B^{oo}_{m,r,s}\, q^{2r+1}\, q^{2s+1}), \qquad (101)\\ \psi^{2\ell} \;&=\; -q^{2\ell+2} \;+\; M^2 q^{2\ell} \;+\\ &\qquad -\,g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(B^{ee}_{-\ell-1,r,s}q^{2r}\, q^{2s} + B^{oo}_{-\ell-1,r,s}q^{2r+1}\, q^{2s+1}),\\ \psi^{2\ell+1} \;&=\; -q^{2\ell+3} \;+\; M^2 q^{2\ell+1} \;-\; 2g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} B^{eo}_{-\ell-1,r,s}\, q^{2r}\, q^{2s+1}, \qquad (102)\end{aligned}$$

where

$$\begin{aligned}B^{eo}_{m,r,s} \;&\equiv\; \sum_{t=0}^{r}\sum_{n=0}^{s}\frac{(-1)^{m+r+s+1}}{\Gamma(m+n+t+2)(r-t)!(s-n)!}\;{}_{2n+2t+1}C_{2t},\\ B^{ee}_{m,r,s} \;&\equiv\; \sum_{t=0}^{r}\sum_{n=0}^{s}\frac{(-1)^{m+r+s+1}}{\Gamma(m+n+t+2)(r-t)!(s-n)!}\;{}_{2n+2t}C_{2t},\\ B^{oo}_{m,r,s} \;&\equiv\; \sum_{t=0}^{r}\sum_{n=0}^{s}\frac{(-1)^{m+r+s}}{\Gamma(m+n+t+3)(r-t)!(s-n)!}\;{}_{2n+2t+2}C_{2t+1}. \qquad (103)\end{aligned}$$

The Hamiltonian (100) is

$$H \;=\; \sum_{m=0}^{\infty} p_m q^{m+1} \;-\; \left(\frac{1}{2}(q^1)^2 + \frac{M^2}{2}(q^0)^2 - \frac{g}{3}(\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{n!})^3\right). \qquad (104)$$

5.1 Perturbative case

When the coupling constant g is zero , the system becomes that of the simple harmonic oscillator, with negative ω^2 ($\omega^2 = -M^2$), discussed in the subsection 3.2. The phase space around the fixed point $q = 0$ is described by (q^0, p_0) and is two dimensional.

If we use the perturbative expansion all the constraints remain in the second class. Then the degrees of freedom will not change and the dimension is equal to 2 [13][15].

$q = 0$ is unstable local maximum. The above 2 degrees of freedom are unstable modes

$$q(t) \quad = \quad a^1 e^{Mt} + a^{-1} e^{-Mt} + O(g). \tag{105}$$

The EL constraints are used iteratively to find solutions in terms of two constants $a^{\pm 1}$. [8] The symplectic two form and the Hamiltonian for this case are

$$\Omega \quad = \quad 2M \; da^1 \wedge da^{-1} \; + \; O(g), \tag{106}$$

$$H = \quad = \quad -2M^2 \; a^1 a^{-1} \; + \; O(g). \tag{107}$$

The a^1 mode describes rolling solution satisfying $q(-\infty) = 0$. Under this boundary condition $Q(\lambda)$ is given in a similar form as in the p-adic particle [4]

$$q(t) = \sum_{n=1}^{\infty} a^n (a^1) e^{nMt} \tag{108}$$

where a^n is determined in terms of a^1 by the recursive relation

$$a_n \quad = \quad \frac{g}{(1-n^2)M^2} \sum_{j=1}^{n-1} e^{-2(n^2-nj+j^2)M^2} \; a_j a_{n-j}, \quad (n \geq 2). \tag{109}$$

Note that the stable and unstable lagrangian submanifold of the fixed point $q = 0$ are both one dimensional.

5.2 Physical space around $q = \frac{M^2}{g}$

We are interested in solving the non-linear constraint of the tachyon particle (99). It has two constant solutions, $q = 0$ and $q = \frac{M^2}{g}$. The first one is corresponding to the perturbative case of the last subsection. Here we examine the perturbation around the second solution.

New canonical coordinates are introduced as

$$\begin{aligned} q^0 \quad &= \quad \frac{M^2}{g} \; + \; \widetilde{q}^0, \qquad q^j \; = \; \widetilde{q}^j, \quad (j > 0) \\ p_{2m+1} \quad &= \quad \widetilde{p}_{2m+1} - \frac{M^4}{g} \frac{(-1)^{m+1}}{(m+1)!}, \qquad p_{2m} \; = \; \widetilde{p}_{2m}. \end{aligned} \tag{110}$$

[8]A numerical analysis of time dependent solutions of string field theory with one arbitrary function was given by Fujita and Hata [6].

In terms of these variables the constraints are

$$
\begin{aligned}
\psi^{2\ell} &= -\widetilde{q}^{2\ell+2} + M^2\widetilde{q}^{2\ell} - 2M^2\sum_{s=0}^{\infty}\frac{(-2)^s\widetilde{q}^{2s+2\ell}}{s!} \\
&\quad -g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(B^{ee}_{-\ell-1,r,s}\ \widetilde{q}^{2r}\ \widetilde{q}^{2s} + B^{oo}_{-\ell-1,r,s}\ \widetilde{q}^{2r+1}\ \widetilde{q}^{2s+1}), \\
\psi^{2\ell+1} &= -\widetilde{q}^{2\ell+3} + M^2\widetilde{q}^{2\ell+1} - 2M^2\sum_{s=0}^{\infty}\frac{(-2)^s\widetilde{q}^{2s+2\ell+1}}{s!} \qquad (111) \\
&\quad -2g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}B^{eo}_{-\ell-1,r,s}\ \widetilde{q}^{2r}\ \widetilde{q}^{2s+1}.
\end{aligned}
\tag{112}
$$

The momentum constraints (101) are

$$
\begin{aligned}
\varphi_{2m} &= \widetilde{p}_{2m} - \widetilde{q}^1\ \delta_{m,0} - 2\ M^2\sum_{s=0}^{\infty}B^{eo}_{m,0,s}\ \widetilde{q}^{2s+1} \qquad (113) \\
&\quad - 2\ g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}B^{eo}_{m,r,s}\ \widetilde{q}^{2r}\ \widetilde{q}^{2s+1}, \\
\varphi_{2m+1} &= \widetilde{p}_{2m+1} + 2M^2\sum_{s=0}^{\infty}B^{ee}_{m,0,s}\ \widetilde{q}^{2s} \\
&\quad + g\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(B^{ee}_{m,r,s}\ \widetilde{q}^{2r}\ \widetilde{q}^{2s} + B^{oo}_{m,r,s}\ \widetilde{q}^{2r+1}\ \widetilde{q}^{2s+1}).
\end{aligned}
\tag{114}
$$

For $g = 0$ the EL constraints (112) have solutions

$$
\widetilde{q}^m \approx \sum_r A^r\ (\kappa_r)^m \tag{115}
$$

where $A^r(t)$'s are arbitrary functions of time and κ_r's are infinite number of solutions of

$$
\kappa^2 - M^2 + 2M^2e^{-2\kappa^2} = 0. \tag{116}
$$

There are infinite number of complex quartet solutions

$$
\{\ \kappa_N,\ \kappa^*_N,\ -\kappa_N,\ -\kappa^*_N,\ \},\quad N = 1, 2, 3, ... \tag{117}
$$

Two real solutions $\pm\kappa_0 = \pm\sqrt{-\frac{1}{2}\mathrm{ProductLog}[-1, -\frac{1}{2e}]}$ are present only for a value of $M^2 = -\frac{1}{2}(1 + \mathrm{ProductLog}[-1, -\frac{1}{2e}]) \approx 0.916$. In a power series on g the solutions of the EL constraints are given by

$$
\begin{aligned}
\widetilde{q}^m &\approx \sum_r A^r_{(0)}\ (\kappa_r)^m + g\sum_{rs}A^{rs}_{(1)}\ (\kappa_r + \kappa_s)^m + g^2\sum_{rst}A^{rst}_{(2)}\ (\kappa_r + \kappa_s + \kappa_t)^m + \\
&\equiv f^m[A],
\end{aligned}
\tag{118}
$$

where

$$
A^r_{(0)} = A^r, \qquad A^{rs}_{(1)} = \frac{-A^r A^s e^{-2(\kappa_r^2+\kappa_r\kappa_s+\kappa_s^2)}}{(\kappa_r+\kappa_s)^2 - M^2 + 2M^2 e^{-2(\kappa_r+\kappa_s)^2}},
$$

$$
A^{rst}_{(2)} = \frac{2\ A^r A^s A^t e^{-(\kappa_r+\kappa_s+\kappa_t)^2} e^{-\kappa_r^2} e^{-(\kappa_s+\kappa_t)^2} e^{-2(\kappa_s^2+\kappa_s\kappa_t+\kappa_t^2)}}{((\kappa_r+\kappa_s+\kappa_t)^2 - M^2 + 2M^2 e^{-2(\kappa_r+\kappa_s+\kappa_t)^2})((\kappa_s+\kappa_t)^2 - M^2 + 2M^2 e^{-2(\kappa_s+\kappa_t)^2})} \tag{119}
$$

Eqs. (118) and (119) give the solution of the EL equation (97).

The EL constraints can be solved as

$$
\widetilde{q}^m - f^m[A] \approx 0, \qquad (m \geq 0), \tag{120}
$$

where the functions $A^r(t)$ satisfy

$$
\dot{A}^r(t) = \kappa_r\ A^r(t) \tag{121}
$$

as in the p-adic case.

The momentum constraints (114) are solved using (120)

$$
\begin{aligned}
\widetilde{p}_{2m} \ &- \ f^1[A]\ \delta_{m,0} \ - \ 2\ M^2 \sum_{s=0}^{\infty} B^{eo}_{m,0,s}\ f^{2s+1}[A] \\
&- \ 2\ g \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} B^{eo}_{m,r,s}\ f^{2r}[A]\ f^{2s+1}[A] \approx 0, \\
\widetilde{p}_{2m+1} \ &+ \ 2M^2 \sum_{s=0}^{\infty} B^{ee}_{m,0,s}\ f^{2s}[A] \\
&+ \ g \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (B^{ee}_{m,r,s}\ f^{2r}[A]\ f^{2s}[A] + B^{oo}_{m,r,s}\ f^{2r+1}[A]\ f^{2s+1}[A]) \approx 0.
\end{aligned} \tag{122}
$$

Since we have the solutions of the coordinates q^m and momenta p_m in terms of infinite arbitrary constants A^r we can compute the symplectic two form (18) in terms of these constants. The computation is similar to the p-adic particle case and at lowest order in g

$$
\begin{aligned}
\Omega &= \sum_m dq^m \wedge\ dp_m \\
&= \sum_r \kappa_r\ (1 - 4M^2 e^{-2\kappa_r^2})\ dA^r \wedge dA^{-r}\ .
\end{aligned} \tag{123}
$$

Similarly the Hamiltonian in (104)

$$
\begin{aligned}
H \ &= \ \sum_{j=0}^{\infty} \widetilde{p}_j \widetilde{q}^{j+1} \ - \ \frac{M^6}{6g^2} \ - \ \frac{1}{2}(\widetilde{q}^1)^2 \ - \ \frac{M^2}{2}(\widetilde{q}^0)^2 \\
&+ M^2 (\sum_{n=0}^{\infty} \frac{(-1)^n \widetilde{q}^{2n}}{n!})^2 \ + \ \frac{g}{3}(\sum_{n=0}^{\infty} \frac{(-1)^n \widetilde{q}^{2n}}{n!})^3
\end{aligned} \tag{124}
$$

is, at lowest order in g ,

$$H \;=\; -\frac{M^6}{6g^2} \;-\; \sum_r \;(1-4M^2e^{-2\kappa_r^2})\;\kappa_r^2\;A^rA^{-r}. \tag{125}$$

The forms of the symplectic two form Ω (123) and the Hamiltonian H (125) show that this is an infinite dimensional system. This implies that there are infinite continuous excitations around the tachyon vacuum.

So far we are assuming $q(t)$ to be any sufficiently smooth functions. If we are interested in the field $q(t)$ which approaches to the tachyonic vacuum at $t=+\infty$. It restricts the modes with

$$Re(\kappa_r) \;<\; 0. \tag{126}$$

In restricting the function space by imposing (126) there appears no term satisfying $\kappa_s+\kappa_r \;=\; 0$ in the sum in (123). The symplectic two form Ω (123) vanishes identically. Similarly the Hamiltonian H (125) becomes a constant $-\frac{M^6}{6g^2}$, asymptotic value of energy at $t=+\infty$. (They are shown to be valid up to g^2.) They show that the system with the boundary condition is an infinite dimensional lagrangian submanifold. Following Sen's conjecture [21] these excitations around the closed string vacuum may be interpreted as closed string states. This interpretation is not clear among other reasons because the possible closed string states would correspond to the quantization of the above solutions and we have not perform this quantization.

The rolling solution is $q=0$ at $t=-\infty$ and it is expected to pass through $q=\frac{M^2}{g}$ at $t=+\infty$. The initial energy should be $H=0$, which is different from the above non-perturbative one with energy $H=-\frac{M^6}{6g^2}$. Since we have a conservation of energy within our model this is not possible, it means that the tachyon will not stop at the minimum but it will oscillate as in the p-adic case. See fig. 2 for a summary of string field theory results.

As in the p-adic case, if we evaluate the energy without imposing any boundary condition the solutions do not have a definite sign of the energy as is seen in Appendix C.

6 Discussions

The 1+1 dimensional Hamiltonian formulation of non-local theories is a valuable tool to construct the physical reduced phase space of a non-local theory. The idea is similar to the construction of the physical phase space of gauge theories.

We have shown that there are two possible physical phase spaces, perturbative and non-perturbative around fixed points, these dimensions do not coincide in general. In particular for the case of the p-adic particle around the fixed point $q=0$ the physical space is zero dimensional at perturbative level. There are no excitation modes around this tachyon vacuum. For the fixed point $q=\frac{1}{g}$ the perturbative phase space is infinite dimensional. The solutions contain an infinite number of constants with a non-vanishing symplectic structure. For the rolling type solutions the physical space is infinite dimensional lagrangian submanifold instead.

In the case of string field theory at lowest truncation level the perturbative phase space around $q = 0$ is two dimensional, which coincides with the dimension of the free theory, and for the non-trivial fixed point $q = \frac{M^2}{g}$ it is perturbatively infinite dimensional. There are continuous excitations around the tachyon vacuum.

The manifold of the solutions for the p-adic particle and string field theory case is different. Around local maxima the space of solutions of the p-adic particle is infinite dimensional while string field is two dimensional. Around local minima (tachyonic vacua) the physical excitations for the p-adic particle is zero dimensional while string field is infinite dimensional. The different phase space structure is due to the dispersion relation for the solutions we found, see eq (83) and eq (116).

It is also noted that in both cases, p-adic and string field theory, the solutions obtained without imposing any boundary condition have infinite dimensional physical phase space and do not have a definite sign for the energy, which in addition, is not bounded from below.

It would be interesting to examine solutions describing the tachyon condensation in these models by introducing dissipation of energy suitably.

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A Harmonic oscillator

The Hamiltonian and constraints of simple harmonic oscillator are, (32)

$$
\begin{aligned}
H &= p_m q^{m+1} - \frac{1}{2}(q^1)^2 + \frac{w^2}{2}(q^0)^2 && (127)\\
\varphi_0 &= p_0 - q^1 \approx 0, \qquad \varphi_j = p_j \approx 0, \quad (j > 0), && (128)\\
\psi^m &= q^{m+2} + w^2 q^m \approx 0, \quad (m \geq 0). && (129)
\end{aligned}
$$

EL constraint (129) is expressed as

$$
\begin{pmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \\ .. \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 & 1 & ... & & & \\ 0 & \omega^2 & 0 & 1 & ... & & \\ 0 & 0 & \omega^2 & 0 & 1 & ... & \\ 0 & 0 & 0 & \omega^2 & 0 & 1 & ... \\ ... & & & & & & \end{pmatrix} \begin{pmatrix} q^0 \\ q^1 \\ q^2 \\ q^3 \\ .. \end{pmatrix} \tag{130}
$$

The coefficient matrix $\mathcal{D}$ has two zero vectors

$$\begin{pmatrix} 1 \\ (iw) \\ (iw)^2 \\ (iw)^3 \\ .. \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ (-iw) \\ (-iw)^2 \\ (-iw)^3 \\ .. \end{pmatrix} \tag{131}$$

Then the general solution of the linear constraints (129) is

$$q^m(t) \ - \ A(t)(iw)^m + A^*(t)(-iw)^m \approx 0. \tag{132}$$

where the functions $A(t)$, appearing as coefficients of the null vectors, should satisfy

$$\dot{A}(t) = i\omega A(t) \tag{133}$$

in order (132) is consistent with the Hamilton's equation $\frac{d}{dt}q^m = q^{m+1}$.

The momentum constraints are solved by

$$\begin{aligned} &p_0 - (A(t)(iw) + A^*(t)(-iw)) \approx 0 \\ &p_n \ \approx \ 0, \ (n > 0) \end{aligned} \tag{134}$$

All phase space variables (q^m, p_m) are described in terms of (A, A^*).

Note the Hamiltonian (127), using the expressions (132) and (134), becomes

$$H \ = \ 2\omega^2 A^* A. \tag{135}$$

The symplectic two form Ω of this system is

$$\Omega \ = \ \sum_{m=0} dp_m \wedge dq^m \ = \ 2i\omega \ dA^* \wedge dA. \tag{136}$$

They show (A, A^*) plays a role of the canonical pair with dimension two.

B Energy of the p-adic particle

We have shown the p-adic particle around the non-perturbative fixed point $g = \frac{1}{g}$ is described by the Hamiltonian (93)

$$H \ = \ \frac{1}{6g^2} \ - \ 2 \sum_r \kappa_r^2 \ A^r A^{-r} \tag{137}$$

with the canonical form (92)

$$\Omega \ = \ \sum_r 2\kappa_r \ dA^r \wedge dA^{-r}, \tag{138}$$

where sums are taken over all modes of solutions of (82), $e^{\kappa^2} - 2 = 0$. There are infinite number of complex quartet solutions

$$\{ \ \kappa_N, \ \overline{\kappa_N}, \ \ -\kappa_N, \ \ -\overline{\kappa_N}, \ \}, \quad N = 0, 1, 2, 3, ... \tag{139}$$

where $\kappa_N = a_N + ib_N$, $a_N^2 - b_N^2 = \log\, 2$, $2a_N b_N = 2N\pi$, $a_N > 0$, $b_N \geq 0$. For $N = 0$, $b_0 = 0$ then there are only two real solutions $\pm\kappa_0$.We use these modes in the symplectic form Ω (138) and the Hamiltonian (137) explicitly they become

$$\begin{aligned}
\Omega &= 4\sum_{N>0} [\kappa_N\, dA^N \wedge dA^{-N} + \overline{\kappa_N}\, d\overline{A^N} \wedge d\overline{A^{-N}}] + 4\kappa_0\, dA^0 \wedge dA^{-0} \\
H &= \frac{1}{6g^2} - 4\sum_{N>0}\left[{\kappa_N}^2\, A^N A^{-N} + \overline{\kappa_N}^2\, \overline{A^N}\, \overline{A^{-N}}\right] - 4\kappa_0^2\, A^0 A^{-0}.
\end{aligned} \tag{140}$$

Now, if we write the variables in terms of real coordinates as

$$A^N = E_N + iB_N, \quad A^{-N} = C_N + iD_N, \quad A^0 = E_0, \quad A^{-0} = C_0, \tag{141}$$

the symplectic form Ω tells that the canonical momenta conjugate to C and D are

$$P_{C_N} = 8(a_N E_N - b_N B_N), \quad P_{D_N} = 8(-a_N B_N - b_N E_N), \quad P_{C_0} = 4a_0 E_0. \tag{142}$$

Then we obtain

$$\Omega = \sum_{N>0} [dP_{C_N} \wedge dC_N + dP_{D_N} \wedge dD_N] + dP_{C_0} \wedge dC_0, \tag{143}$$

$$H = \frac{1}{6g^2} - \sum_{N>0}\left[(P_{C_N}, P_{D_N})\begin{pmatrix} a_N & -b_N \\ b_N & a_N \end{pmatrix}\begin{pmatrix} C_N \\ D_N \end{pmatrix}\right] - a_0 P_{C_0} C_0. \tag{144}$$

Since the C, D and their conjugate P_C, P_D are real canonical variables the Hamiltonian is not bounded from below. Using symplectic transformations the Hamiltonian can further be simplified.

Using a symplectic transformation

$$C = \frac{1}{\sqrt{2}}(q + p'), \quad P_C = \frac{1}{\sqrt{2}}(p - q'), \tag{145}$$

$$D = \frac{1}{\sqrt{2}}(q' + p), \quad P_D = \frac{1}{\sqrt{2}}(p' - q), \tag{146}$$

the Hamiltonian (144) becomes

$$\begin{aligned}
H = \frac{1}{6g^2} &+ \frac{1}{2}\sum_{N>0}\left[(p, p')\begin{pmatrix} b & -a \\ -a & -b \end{pmatrix}\begin{pmatrix} p \\ p' \end{pmatrix} + (q, q')\begin{pmatrix} b & a \\ a & -b \end{pmatrix}\begin{pmatrix} q \\ q' \end{pmatrix}\right]_N \\
&+ \frac{a}{2}(p_0^2 - q_0^2).
\end{aligned} \tag{147}$$

After rotation of (q, q'), then rescaling the coordinates, it follows

$$\begin{aligned}
H = \frac{1}{6g^2} &+ \frac{1}{2}\sum_{N>0}[(\tilde{p}, \tilde{p}')\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \tilde{p} \\ \tilde{p}' \end{pmatrix} \\
&+ (\tilde{q}, \tilde{q}')\begin{pmatrix} -(a^2 - b^2) & 2ab \\ 2ab & (a^2 - b^2) \end{pmatrix}\begin{pmatrix} \tilde{q} \\ \tilde{q}' \end{pmatrix}]_N + \frac{1}{2}(\tilde{p}_0^2 - a_0^2\tilde{q}_0^2).
\end{aligned} \tag{148}$$

The p^2 term and ${p'}^2$ term have opposite signs. There remains a freedom of $SO(1,1)$ rotation, which keeps kinetic terms invariant. However it can not eliminate the $\tilde{q}\tilde{q}'$ term and the Hamiltonian is not diagonalized completely.

Eigen frequencies are given from

$$\frac{d^2}{dt^2}\begin{pmatrix} \tilde{q} \\ \tilde{q}' \end{pmatrix} = \begin{pmatrix} (a^2-b^2) & -2ab \\ 2ab & (a^2-b^2) \end{pmatrix}\begin{pmatrix} \tilde{q} \\ \tilde{q}' \end{pmatrix} \tag{149}$$

The eigenvalues of the matrix are $(a_N \pm ib_N)^2 = \kappa_N^2$ in agreement with the solution (86) and (87). Summing up, it can be seen that the non locality shows indefinite signs of the energy. This agrees with the general discussion of the Ostrogradski [14] in [13].

C Energy for the String Field Theory

We can analyze the string field theory case as in the p-adic particle. Now we consider the local minimum $q = \frac{M^2}{g}$ (tachyonic vacuum) rather than the local maximum. The symplectic form is (123)

$$\Omega = \sum_r \kappa_r \left(1 - 4M^2 e^{-2\kappa_r^2}\right) dA^r \wedge dA^{-r} \ . \tag{150}$$

and the Hamiltonian is (125)

$$H = -\frac{M^6}{6g^2} - \sum_r \left(1 - 4M^2 e^{-2\kappa_r^2}\right) \kappa_r^2 \ A^r A^{-r}, \tag{151}$$

where κ_r's are solutions of (116), $\kappa^2 - M^2 + 2M^2 e^{-2\kappa^2} = 0$. It have an infinite number of complex quartet solutions

$$\{\ \kappa_N,\ \overline{\kappa_N},\ \ -\kappa_N,\ \ -\overline{\kappa_N},\ \},\quad N = 1, 2, 3, ... \tag{152}$$

The symplectic form Ω (150) and the Hamiltonian (151) are

$$\begin{aligned} \Omega &= \sum_{N>0} [\kappa_N(1 - 4M^2 e^{-2\kappa_{N^2}})\ dA^N \wedge dA^{-N} + \overline{\kappa_N(1 - 4M^2 e^{-2\kappa_{N^2}})}\ d\overline{A^N} \wedge d\overline{A^{-N}}] \\ H &= -\frac{M^6}{6g^2} - 2\sum_{N>0}\left[\kappa_N{}^2(1 - 4M^2 e^{-2\kappa_{N^2}})A^N A^{-N} + \overline{\kappa_N^2(1 - 4M^2 e^{-2\kappa_{N^2}})}\ \overline{A^N}\ \overline{A^{-N}}\right]. \end{aligned} \tag{153}$$

Now we write them in terms of real coordinates as

$$A^N = E_N + iB_N, \quad A^{-N} = C_N + iD_N, \tag{154}$$

the symplectic form Ω tells that the canonical momenta conjugate to C and D are

$$P_{C_N} = 8(\tilde{a}_N E_N - \tilde{b}_N B_N), \quad P_{D_N} = 8(-\tilde{a}_N B_N - \tilde{b}_N E_N), \tag{155}$$

where

$$\kappa_N = a_N + ib_N,\ \ \kappa_N^2(1 - 4M^2 e^{-2\kappa_N^2}) \equiv 2(\tilde{a}_N + i\tilde{b}_N). \tag{156}$$

Then we obtain

$$
\begin{aligned}
\Omega &= \sum_{N>0} [dP_{C_N} \wedge dC_N + dP_{D_N} \wedge dD_N], && (157)\\
H &= -\frac{M^6}{6g^2} - \sum_{N>0} \left[(P_{C_N}, P_{D_N}) \begin{pmatrix} a_N & -b_N \\ b_N & a_N \end{pmatrix} \begin{pmatrix} C_N \\ D_N \end{pmatrix} \right]. && (158)
\end{aligned}
$$

They have same forms as the p-adic particle case. The C, D and their conjugate P_C, P_D are real canonical variables therefore the Hamiltonian is not bounded from below.

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